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Beyond Logical Structure of Quantum Mechanics: An Algebra of Projection Operators in Quantum Field Theory

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Abstract An algebra is derived, which generates dynamics of the quantum field theory (QFT), based on creation and annihilation operators. We show the existence of a commutative subalgebra which determines microscopic projective processes, and is sufficient to incorporate unitary transformations of all global phenomena.

Keywords Quantum Field Theory · Algebra · Projection Operator · Quantum Logic

Von Neumann's efforts of reconstructing quantum mechanics based on experimentally testable propositions bare fruit as the logic of quantum mechanics [1]. The basic idea of the logic of quantum mechanics is simple: any statement about quantum phenomena, no matter how complicated, should consist of the atomic propositions that its truth value is determined by the experiment. In other words, the whole of quantum mechanics should be explained by the fact that is established experimentally.

Traditionally, the lattice of any closed subspaces of the given Hilbert space is called the logic of quantum mechanics, or simply called quantum logic [6]. Since any closed subspace one-to-one corresponds to the projection onto it, the logic of quantum mechanics is rephrased as the lattice of projections.

However, when it comes to discuss quantum field theory (QFT), such lattice formulation is not applicable. That is, atomic propositions, namely projections, of the logic of quantum mechanics is no more atomic: it is further decomposed into the creation and anihilation operators.

For this reason, we try to find a small basic algebra of projection operators which generates dynamical processes of QFT. Since there are only two types of different operators in the QFT, namely the creation operator $a^\dagger$ and annihilaton operator a, the algebra is appropriate for the foundation of QFT rather than the logic of quantum mechanics.

To begin with let us fix the rule how a particle with information x is created or annihilated from the state vector $|\psi\rangle$ in $\mathcal{H}$. We denote by $|\cdot\rangle$ the state if we have no information. If we know,

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or we can check that there is a quantum field (or simply particle) with information x in the state, we denote it by $|x, \cdot\rangle$. These states can be transformed each other by creating and by annihilating a particle from the states as follows:

$$|x, \cdot\rangle = a_x^\dagger |\cdot\rangle, \qquad |\cdot\rangle = a_x |x, \cdot\rangle. \tag{1}$$

It is possible that the state is either in $|\cdot\rangle$ or in $|x, \cdot\rangle$, but not able to decide. Then the state is given by their super position with α, β being complex numbers:

$$|\psi\rangle = \alpha|\cdot\rangle + \beta|x, \cdot\rangle. \tag{2}$$

We assume that a particle with x does not annihilate from $|\cdot\rangle$, nor create another one to $|x, \cdot\rangle$. We represent this situation as

$$a_x^\dagger |x, \cdot\rangle = 0, \qquad a_x|\cdot\rangle = 0. \tag{3}$$

0's in the right hand sides mean that the states on the left are not physical.

If we define

$$P_x := a_x^\dagger a_x, \qquad P_x^\perp := a_x a_x^\dagger$$

we immediately obtain from (1)

$$P_x|x, \cdot\rangle = |x, \cdot\rangle, \qquad P_x^\perp|\cdot\rangle = |\cdot\rangle. \tag{4}$$

We can say, from these results that P_x and $P_x^\perp$ are operators which check if a particle with information x exists or not, respectively. Applying the operations $P_x^2,\ (P_x^\perp)^2,\ P_x + P_x^\perp$ to $|\psi\rangle$ given by (2) and using (4) we find

$$P_x^2|\psi\rangle = P_x|\psi\rangle, \qquad (P_x^\perp)^2|\psi\rangle = P_x^\perp|\psi\rangle,$$
$$(P_x + P_x^\perp)|\psi\rangle = |\psi\rangle.$$

Because $|\psi\rangle$ is an arbitrary state

$$P_x^2 = P_x, \qquad (P_x^\perp)^2 = P_x^\perp \tag{5}$$
$$P_x + P_x^\perp = 1 \tag{6}$$

must hold. (5) means that P_x and $P_x^\perp$ are projection operators.

Operating $a_x^\dagger a_x^\dagger$ and $a_x a_x$ to $|\psi\rangle$ of (2), and using (1) and (3), we obtain

$$a_x a_x|\psi\rangle = 0, \qquad a_x^\dagger a_x^\dagger|\psi\rangle = 0.$$

Since $|\psi\rangle$ is arbitrary

$$a_x a_x = 0, \qquad a_x^\dagger a_x^\dagger = 0 \tag{7}$$

must hold irrespect to the state. These relations are called the exclusion principle of Fermionic particles in the QFT.

It is apparent that the operators a_x and $a_x^\dagger$ exchange each other under the time reversal. In order to make it more complete we introduce conjugate states $\langle\psi|$ by

$$\langle\cdot, x| = \langle\cdot|a_x, \qquad \langle\cdot| = \langle\cdot, x|a_x^\dagger. \tag{8}$$

It follows from (3)

$$\langle\cdot|a_x^\dagger = 0, \qquad \langle\cdot, x|a_x = 0.$$

We call this symmetry a creation annihilation duality, which we represent symbolically $a_x \rightleftharpoons a_x^\dagger,\ |\psi\rangle \rightleftharpoons \langle\psi|$. If we define the inner product of the states $|\psi\rangle$ and $\langle\psi'|$ by $\langle\psi'|\psi\rangle$, we find immediately

$$\langle\cdot|x,\cdot\rangle = \langle\cdot,x|\cdot\rangle = 0 \tag{9}$$

from (3).

Now we consider the cases in which more than two independent information, say x and y, are included. We then define, in the case $x \neq y$,

$$a_x a_y = -a_y a_x, \quad a_x^\dagger a_y^\dagger = -a_y^\dagger a_x^\dagger, \quad a_x a_y^\dagger = -a_y^\dagger a_x.$$

The minus signs on the right hand sides are due to the fact such that in the case $x = y$ the exclusion principle (7) holds as well. When there are two independent information the general physical state is of the form

$$|\psi\rangle = \alpha|\cdot\rangle + \beta|x,\cdot\rangle + \gamma|y,\cdot\rangle + \delta|x,y,\cdot\rangle \tag{10}$$

with $\alpha,\beta,\gamma,\delta$ being arbitrary complex numbers.

In addition to P_x and P_y there is another bilinear combination of the creation and annihilation operators, namely $a_y^\dagger a_x$, which changes the state $|x,\cdot\rangle$ to $|y,\cdot\rangle$. In the QFT every physical process can be decomposed into summation of products of only two types of bilinear operators either $a_y^\dagger a_x$ with $x \neq y$, or $a_x^\dagger a_x = P_x$. Therefore these are sufficient to formulate all physical processes in the QFT. An important observation in our argument is that the operator $P_{y\leftrightsquigarrow x}$ defined by

$$P_{y\leftrightsquigarrow x} := a_y^\dagger U_{yx} a_x + a_x a_x^\dagger \tag{11}$$

is a projection operator satisfying [7]

$$P_{y\leftrightsquigarrow x}^2 = P_{y\leftrightsquigarrow x}$$

for any c-number function U_{yx}. If $U_{xx} = 1$, (11) becomes $P_{x\leftrightsquigarrow x} = 1$ because of (6). Let us call $P_{y\leftrightsquigarrow x}$ the transition operator of the state from x to y. In fact if we apply $P_{y\leftrightsquigarrow x}$ to $|\psi\rangle$ of (10), we find

$$P_{y\leftrightsquigarrow x}|\psi\rangle = \alpha|\cdot\rangle + U_{yx}\beta|y,\cdot\rangle + \gamma|y,\cdot\rangle.$$

Notice that only information x is changed into y but leaving all other information unchanged. The missing of the δ term is reasonable because of the exclusion principle.

The main contribution of this paper is a discovery of a closed algebra formed by the projection operators $P_{y\leftrightsquigarrow x}$ and P_x, when the function U_{yx} satisfies the connectivity condition

$$U_{zy}U_{yx} = U_{zx}. \tag{12}$$

Using the notation $[A,B] := AB - BA$, it is given by

$$[P_x, P_w] = 0, \qquad \text{for all } x, w$$

$$\begin{aligned}[P_{y\leftrightsquigarrow x}, P_w] &= P_{y\leftrightsquigarrow x} + P_x - 1, \\ &\qquad \text{if } \ w = x, \ \text{for all } x,y,z \\ &= -P_{y\leftrightsquigarrow x} - P_x + 1, \\ &\qquad \text{if } \ w = y, \ \text{for all } x,y,z\end{aligned}$$

$$
\begin{aligned}
[P_{y\leftrightsquigarrow x}, P_{z\leftrightsquigarrow w}] &= -P_{y\leftrightsquigarrow x} + P_{z\leftrightsquigarrow x}, \\
&\qquad \text{if } w = x, \text{ for all } x, y, z \\
&= P_{y\leftrightsquigarrow x} - P_{z\leftrightsquigarrow x}, \\
&\qquad \text{if } \ w = y, \text{ for all } x, y, z \\
&= 0, \quad \text{if } \ z = y, \text{ for all } x, y, w,
\end{aligned} \tag{13}
$$

which we call the projection algebra PA in the QFT.

This algebra PA is sufficient to determine local properties of dynamics of the QFT, hence all microscopic behaviour of quantum phenomena. We can derive global view of all physical processes by the exponentiation of this algebra according to the well established method in the QFT. In other words the projection operators are the generators of global dynamics. From this point of view it is remarkable that the function U_{yx} in (11) does not appear explicitly in the algebra PA. Hence it does not play any role in generating projective dynamical processes.

When an algebra is given, which generates the dynamics of a physical system, it is important to know a commutative subalgebra of generators, in order to specify the system. Upon some investigations of (13) we find a set of three operators $P_{y\wedge x}, P_{y\vee x}, P_{y\leftrightsquigarrow x}$, defined by

$$
\begin{aligned}
&P_{y\wedge x} := P_y P_x, \quad P_{y\vee x} := P_x + P_y - P_y P_x, \\
&P_{y\leftrightsquigarrow x} := a_y^\dagger U_{yx} a_x + a_x a_x^\dagger, \quad \text{for all } x, y.
\end{aligned} \tag{14}
$$

They are not only commutative with each other

$$
[P_{y\wedge x}, P_{y\vee x}] = [P_{y\vee x}, P_{y\leftrightsquigarrow x}] = [P_{y\leftrightsquigarrow x}, P_{y\wedge x}] = 0, \tag{15}
$$

but also projective by themselves

$$
P_{y\wedge x}^2 = P_{y\wedge x}, \qquad P_{y\vee x}^2 = P_{y\vee x}, \qquad P_{y\leftrightsquigarrow x}^2 = P_{y\leftrightsquigarrow x}.
$$

By applying these operators to the state (10) we obtain

$$
P_{y\wedge x}|\psi\rangle = \delta|x, y, \cdot\rangle, \quad P_{y\vee x}|\psi\rangle = \alpha|x, \cdot\rangle + \beta|y, \cdot\rangle + \delta|x, y, \cdot\rangle.
$$

It is interesting to notice that $P_{y\wedge x}$ and $P_{y\vee x}$ are extended as the most fundamental elements of the commutative lattice. That is, if the binary relations $\wedge$ (meet) and $\vee$ (join) on the set of projections is defined by

$$
P \wedge P' := PP' \text{ and } P \vee P' := P + P' - PP'
$$

for any projections P and P', we obtain $P_{y\wedge x} = P_y \wedge P_x$ and $P_{y\vee x} = P_y \vee P_x$.

However, only $P_{y\leftrightsquigarrow x}$ cannot be extended as is the case of $P_{y\wedge x}$ and $P_{y\vee x}$, because $P_{y\leftrightsquigarrow x}$ is defined based on the non-projection operator $a_y^\dagger U_{yx} a_x$. In other words, $P_{y\leftrightsquigarrow x}$ is not captured by the conventional lattice theory.

If the operators of (14) are all self-adjoint, the subalgebra is called the Cartan subalgebra. However this is not our case because $P_{y\leftrightsquigarrow x}$ changes the state $|x, \cdot\rangle$ to $|y, \cdot\rangle$, hence is not Hermite. Nevertheless the set of operators (14) will play a central role when the algebra is exponentiated to represent global phenomena.

We now turn our attention to the global view, so that our consideration is not constrained to two microscopic level states of (10). Once we incorporate other states like $|z, \cdot\rangle$ into account, the operator $P_{z\leftrightsquigarrow y}$ does not commute with those of (14), hence the subalgebra (15) must be extended to the PA.

In order to see how it works let us consider a transition of a particle in the initial state $|x_0, \cdot\rangle$ to other state $|x_n, \cdot\rangle$ after n steps. We may specify U by a gauge field $\mathcal{A}(x)$ according to

$$U_{yx} = \exp\left(i\int_x^y \mathcal{A}(x')dx'\right)$$

which satisfies the connectivity condition (12). Using the PA we can show

$$[P_{z\leftrightsquigarrow x}, P_{z\leftrightsquigarrow y}P_{y\leftrightsquigarrow x}] = 0.$$

After a simple manipulation we find

$$\begin{aligned} &\langle \cdot, x_n|P_{x_n\leftrightsquigarrow x_{n-1}} \cdots P_{x_2\leftrightsquigarrow x_1}P_{x_1\leftrightsquigarrow x_0}|x_0, \cdot\rangle \\ &= \langle \cdot|U_{x_n x_{n-1}} \cdots U_{x_2 x_1}U_{x_1 x_0}|\cdot\rangle = U_{x_n x_0} \end{aligned} \tag{16}$$

showing that an ordered sequence of transitions yields a unitary transformation from $|x_0, \cdot\rangle$ to $|x_n, \cdot\rangle$.

Usually it is said that there are two distinct dynamical processes in quantum mechanics, namely the projection and the unitary transformation. The latter nature is described by the Schrödinger equation. It is, therefore, remarkable that the unitary transformation is incorporated into the projection operator quite naturally, in our approach, owing to the PA.

To be more general we extend (10) to study a state of n particles given by

$$\begin{aligned} |\psi\rangle &= \sum_{i_1,i_2,\cdots,i_n}^{\{0,1\}} \alpha_{i_1,i_2,\cdots,i_n}|i_1, i_2, \cdots, i_n, \cdot\rangle \\ &= \sum_{i_1,i_2,\cdots,i_n} \alpha_{i_1,i_2,\cdots,i_n}(a_1^\dagger)^{i_1}(a_2^\dagger)^{i_2}\cdots(a_n^\dagger)^{i_n}|\cdot\rangle \end{aligned} \tag{17}$$

Here i_k takes values either 1 or 0 for each $k \in \{1, 2, \cdots, n\}$. For simplicity we will use, hereafter, the notation $I := \{i_1, i_2, \cdots, i_n\}$ to represent all possible sets of n combinations of 0 or 1. α_I is an arbitrary constant which fixes the weight of state for every choice of I. There are 2^n such cases, which we have to sum up over all combinations in $\sum_I$. Then (17) is simply written as $|\psi\rangle = \sum_I \alpha_I|I, \cdot\rangle$.

We notice that

$$P_I := (P_1)^{i_1}(P_1^\perp)^{1-i_1}(P_2)^{i_2}(P_2^\perp)^{1-i_2}\cdots(P_n)^{i_n}(P_n^\perp)^{1-i_n}$$

is a projection operator. In fact, if $J = \{j_1, j_2, \cdots, j_n\}$, it satisfies

$$P_J|I, \cdot\rangle = \begin{cases} |I, \cdot\rangle, \text{ if } \quad i_k = j_k, \quad k = 1, 2, \cdots, n \\ 0, \qquad \text{othewise.} \end{cases}$$

In other words P_I projects $|\psi\rangle$ to the particular state $|I, \cdot\rangle$.

Among projections, there are two different types of change of the state $|\psi\rangle$. One is to change the weight α_I in (17) dependent on I. It will be done if we operate, say $\sum_I \beta_I P_I$, to $|\psi\rangle$, from which we obtain

$$\sum_I \beta_I P_I : \quad |\psi\rangle \to |\psi'\rangle = \sum_I \beta_I \alpha_I |I, \cdot\rangle$$

The second type of the change of state takes place by exchange particles from one place to another. In order to simplify the argument we assume a set of first k particles $I_k = \{i_1, i_2, \cdots, i_k\}$ change to other set in the complement $I_k^\perp$. Then the operator

$$P_{\varphi \leftrightsquigarrow \{I_k\}} := \sum_{I_k} \varphi_{I_k}(a_{k+1}^\dagger, \cdots, a_n^\dagger) a_1^{i_1} a_2^{i_2} \cdots a_k^{i_k} \times (P_1^\perp)^{1-i_1} (P_2^\perp)^{1-i_2} \cdots (P_k^\perp)^{1-i_k} + P_1^\perp P_2^\perp \cdots P_k^\perp$$

transfoms the states I_k to those of $I_k^\perp$, according to

$$\sum_{I_k} \alpha_{I_k} |I_k, \cdot\rangle \to \sum_{I_k} \alpha_{I_k} \varphi_{I_k} |\cdot\rangle.$$

Here $\varphi_{I_k}(a_{k+1}^\dagger, \cdots, a_n^\dagger)$ is an arbitrary polynomial function of $a_{k+1}^\dagger, \cdots, a_n^\dagger$ in $I_k^\perp$. This is again a projection operator, satisfying

$$P_{\varphi \leftrightsquigarrow \{I_k\}}^2 = P_{\varphi \leftrightsquigarrow \{I_k\}}.$$

Some examples are in order:

- $\varphi_x(a_y^\dagger, a_z^\dagger) = a_y^\dagger + a_z^\dagger$
$$P_{y+z \leftrightsquigarrow x} : \quad |x, \cdot\rangle \to |y, \cdot\rangle + |z, \cdot\rangle,$$
- $\varphi_x(a_y^\dagger, a_z^\dagger) = a_y^\dagger a_z^\dagger$
$$P_{yz \leftrightsquigarrow x} : \quad |x, \cdot\rangle \to |y, z, \cdot\rangle$$
,
- $\varphi_{x+y}(a_z^\dagger, a_w^\dagger) = a_z^\dagger a_w^\dagger$
$$P_{zw \leftrightsquigarrow x+y} : \quad |x, \cdot\rangle + |y, \cdot\rangle \to |z, w, \cdot\rangle.$$

To conclude, we would like to mention the mico-macro correspondence. As is well known a microscopic transition between states takes place projectively in quantum physics, as well as in the QFT. On the other hand the global behavior of quantum states is determined by unitary transformation, so that the Schrödinger equation is satisfied. If we must incorporate these two different views to a physical system, how they could be consistent each other? This question has been discussed repeatedly since the foundation of quantum mechanics [2].

Our answer to this question is rather simple. Our projective operator $P_{y \leftrightsquigarrow x}$ of (11) which executes the microscopic transition from x to y has a freedom to incorporate internal (or gauge) symmetry U dependent on x and y. In global picture all creation and annihilation operators disappear upon taking an inner product of states, such that only freedoms of unitarity are left. Therefore projective nature in microscopic level is naturally turned to unitary one. An important fact is that when operators is replaced by unitary functions the connectivity condition (12) is fulfilled, as we have shown in some example (16). This means that nonanalytic feature of projective transformation in microscopic level is changed to a smooth function, hence explains how the transition between micro and macro physics undergoes.

Recently there have been some interesting arguments of the micro-macro correspondence, within the framework of quantum mechanics, based on category theory [3,5]. We can also see our formulation of the QFT from the view point of category theory. In our case the projection operator $P_{y \leftrightsquigarrow x}$, consisting of a product of creation and annihilation operators, plays the role of a morphism which connects one physical state to another. We have shown that this operation can be generalized to arbitrary number of states. Moreover, since the projective operators form the algebra (13), it is straightforward to see global behavior of the system simply by exponentiation of the algebra. As far as we focus our attention to global behavior of transition amplirtudes we are able to describe physical phenomena in terms of general languages, without reference to microscopic processes.

There is also a work on categorical quantum mechanics of the QFT [4]. Since a harmonic oscillator model was studied in the work, hence discussing Bosonic fields, it does not share our results which are derived from Fermionic nature of fields.

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